Cima *Complex Anal Synerg* (2016) 2:3
DOI 10.1186/s40627-016-0007-9

Complex Analysis and its Synergies

RESEARCH Open Access

A type of Volterra operator

Joseph A. Cima*

*Correspondence:
cima@math.unc.edu
University of North Carolina
at Chapel Hill, Chapel Hill,
USA

Abstract

In Diamantopoulos and Siskakis (Studia Math 140:191–198, 2000), the authors study the action of the classical Cesaro matrix C on the Taylor coefficients of analytic functions on the Hardy spaces $H^p(\mathbb{D})$, $1 < p < \infty$. They convert the matricial action of C on sequences into a Volterra type integral operator $\mathbb{H}$ on H^p. They show that it is bounded for $1 < p < \infty$ and derive estimates on the operator norm of $\mathbb{H}$. We continue this study and show that $\mathbb{H}$ maps boundedly from $H^1(\mathbb{D})$ into the space of Cauchy transforms of finite Borel measures on unit circle. We show that $\mathbb{H}$ is one to one on $H^2(\mathbb{D})$.

Keywords: Hardy spaces, Volterra type operators

Mathematics Subject Classification: 30J10, 47A66

1 Introduction and definitions

We use $\mathbb{D}$ as the notation for the unit disk in the complex plane and $\mathbb{T}$ as its boundary. For $\zeta = \exp(\imath\theta)$, we denote the normalized Lebesgue measure on $\mathbb{T}$ by $\mathrm{d}\mu(\theta) = \mathrm{d}\theta/2\pi$. The classical Hardy space on $\mathbb{D}$ is written as $H^p(\mathbb{D})$ and consists of those analytic functions f for which

$$\sup_{r<1} \left(\int_{\mathbb{T}} |f(r\exp(\imath\theta))|^p \mathrm{d}\mu(\theta) \right) < \infty.$$

For $p \geq 1$, taking the p-th root of this sup yields a norm and with this norm $H^p(\mathbb{D})$ is a Banach space. When $p = 2$, this is Hilbert space and identifying the Taylor coefficients of such an f with a sequence we obtain an isometry from $H^2(\mathbb{D})$ onto the classical sequence space l^2. In addition, there is a natural identification of functions in $H^p(\mathbb{D})$ with functions in a closed subspace of $L^p(\mathbb{T})$ by way of the non-tangential limits of functions f in $H^p(\mathbb{D})$ at points of $\mathbb{T}$, and it shall be clear in our discussion whether we are dealing with points in $\mathbb{D}$ or with points in $\mathbb{T}$. See [1] for more details.

2 The main result

In [2], the authors study the integral operator

$$(\mathcal{H}f)(z) = \int_0^1 \frac{f(t)}{1 - zt} \mathrm{d}t$$

on the classical Hardy spaces H^p of the open unit disk $\mathbb{D}$ [1]. Making this operator well defined is the integrability of $f \in H^p$ on the interval [0, 1] which follows from the well-known Fejer–Riesz inequality [1, p. 46]

$$\int_{-1}^{1} |f(t)|^p \mathrm{d}t \leq \frac{1}{2}\int_0^{2\pi} |f(e^{i\theta})|^p \mathrm{d}\theta, \quad f \in H^p. \tag{1}$$

In [2], they prove that when $p \in (1, \infty)$ the operator $\mathcal{H}$ maps H^p to itself. This is no longer the case when $p = 1$ or $p = \infty$.

The purpose of this paper is to discuss the range of $\mathcal{H}$ on H^1. The main result is the following.

Theorem 1 *The operator $\mathcal{H}$ maps H^1 to the space of Cauchy transforms of measures on the unit circle $\partial\mathbb{D}$. Furthermore, $\mathcal{H}$ is injective*

The proof of this theorem needs a few facts about Cauchy transforms which can be found in [3]. Let M be the Banach space of finite complex Borel measures on $\partial\mathbb{D}$ endowed with the total variation norm $\|\mu\|$ and define

$$\mathscr{C} := \left\{ (C\mu)(z) := \int_{\partial\mathbb{D}} \frac{\mathrm{d}\mu(e^{i\theta})}{1 - e^{-i\theta}z} : \mu \in M \right\},$$

to be the space of Cauchy transforms. This is a Banach space of analytic functions on $\mathbb{D}$ when given the norm

$$\|C\mu\| := \inf\{\|\nu\| : C\mu = C\nu\}.$$

It is well known [3, Ch. 4] that $\mathcal{A}$ (the disk algebra), the space of continuous functions on $\overline{\mathbb{D}}$ which are analytic on $\mathbb{D}$, endowed with the supremum norm $\|g\|_\infty = \sup\{|g(z)| : z \in \overline{\mathbb{D}}\}$, can be identified with the pre-dual of $\mathscr{C}$ via the (Cauchy) pairing

$$(g, C\mu) := \lim_{r\to 1} \int_0^{2\pi} g(re^{i\theta})\overline{C\mu(re^{i\theta})}\frac{\mathrm{d}\theta}{2\pi}. \tag{2}$$

Proof of Theorem 1 For $f \in H^1$ and $g \in \mathcal{A}$, observe the following: For $r \in (0, 1)$,

$$\begin{aligned}
\int_0^{2\pi} g(re^{i\theta})\overline{(\mathcal{H}f)(re^{i\theta})}\frac{\mathrm{d}\theta}{2\pi} &= \int_0^{2\pi} g(re^{i\theta})\overline{\left(\int_0^1 \frac{f(t)}{1 - tre^{i\theta}}\mathrm{d}t\right)}\frac{\mathrm{d}\theta}{2\pi} \\
&= \int_0^1 \overline{f(t)}\left(\int_0^{2\pi} \frac{g(re^{i\theta})}{1 - tre^{-i\theta}}\frac{\mathrm{d}\theta}{2\pi}\right)\mathrm{d}t \\
&= \int_0^1 \overline{f(t)}\left(\oint_{\partial\mathbb{D}} \frac{g(r\zeta)}{1 - tr\overline{\zeta}}\frac{\mathrm{d}\zeta}{\zeta 2\pi i}\right)\mathrm{d}t \\
&= \int_0^1 \overline{f(t)}\left(\frac{1}{2\pi i}\oint_{\partial\mathbb{D}} \frac{g(r\zeta)}{\zeta - tr}\mathrm{d}\zeta\right)\mathrm{d}t \\
&= \int_0^1 \overline{f(t)}g(r^2t)\mathrm{d}t.
\end{aligned}$$

Thus, by (1) (making $f \in L^1[0, 1]$) and the fact that $g(rt) \to g(t)$ uniformly on [0, 1], we have

$$\lim_{r\to 1}\int_0^{2\pi} g(re^{i\theta})\overline{(\mathcal{H}f)(re^{i\theta})}\frac{\mathrm{d}\theta}{2\pi} = \lim_{r\to 1}\int_0^1 \overline{f(t)}g(r^2t)\mathrm{d}t = \int_0^1 \overline{f(t)}g(t)\mathrm{d}t$$

exists. Using (1) again, we get

$$\left|\int_0^1 \overline{f(t)} g(t) \mathrm{d}t\right| \leq \|p\|_\infty \cdot \frac{1}{2}\int_0^{2\pi} |f(e^{i\theta})| \mathrm{d}\theta, \tag{3}$$

and so

$$g \mapsto \lim_{r\to 1}\int_0^{2\pi} g(re^{i\theta})\overline{(\mathcal{H}f)(re^{i\theta})}\frac{\mathrm{d}\theta}{2\pi}$$

defines a continuous linear functional on the disk algebra $\mathcal{A}$. The duality in (2) yields $\mathcal{H}f = C\mu$ for some $\mu \in M$.

To show that $\mathcal{H}$ is injective, observe that for $f \in H^1$, a geometric series computation with the definition of $\mathcal{H}$ will show that

$$(\mathcal{H}f)(z) = \sum_{n=0}^{\infty} z^n \left(\int_0^1 t^n f(t) \mathrm{d}t\right).$$

If $\mathcal{H}f \equiv 0$, then

$$\int_0^1 p(t) f(t) \mathrm{d}t = 0$$

for all polynomials p. The Weisrstrass approximation theorem and the fact that $f \in L^1[0,1]$ yields

$$\int_0^1 f(t) g(t) \mathrm{d}t = 0$$

for all continuous functions g on [0, 1]. By the Riesz representation theorem for the space of finite Borel measures on [0, 1], we conclude that $f(t) = 0$ almost everywhere on [0, 1]. The identity theorem for analytic functions implies that $f \equiv 0$. □

Remark 1 It is well known that when $p \in (1,\infty)$, the dual of H^p can be identified with H^q, where q is the Holder conjugate index for p, via the pairing

$$\begin{aligned}(g, f) &= \lim_{r\to 1}\int_0^{2\pi} g(re^{i\theta})\overline{f(re^{i\theta})}\frac{\mathrm{d}\theta}{2\pi}\\ &= \int_0^{2\pi} g(e^{i\theta})\overline{f(e^{i\theta})}\frac{\mathrm{d}\theta}{2\pi}, \quad g \in H^q, f \in H^p.\end{aligned}$$

With this in mind, one can, for $f \in H^p$, $p \in (1,\infty)$, repeat the proof of Theorem 1, replacing (3) with the estimate

$$\left|\int_0^1 \overline{f(t)} g(t) \mathrm{d}t\right| \lesssim \left(\int_0^{2\pi} |g(e^{i\theta})|^q \mathrm{d}\theta\right)^{1/q}\left(\int_0^{2\pi} |f(e^{i\theta})|^p \mathrm{d}\theta\right)^{1/p}.$$

(note two uses of the Fejer–Riesz inequality along with Holder's inequality), to see that $\mathcal{H}f \in H^p$. This yields an alternate proof of this fact from [2].

Remark 2 One can prove a bit more here. Indeed, if $f \in H^1$, then $f \in L^1[0,1]$, and thus given $\epsilon > 0$, there is a $\delta > 0$, so that

$$\int_{1-\delta}^{1} |f(t)|\mathrm{d}t < \epsilon.$$

For any $\theta \in [0, 2\pi]$ and $r \in (0, 1)$

$$\begin{aligned}(1-r)|\mathcal{H}f(re^{i\theta})| &\leq (1-r)\int_0^1 \frac{|f(t)|}{|1-tre^{i\theta}|}\mathrm{d}t \\ &\leq (1-r)\int_0^{1-\delta} \frac{|f(t)|}{1-(1-\delta)r}\mathrm{d}t + \int_{1-\delta}^{1} |f(t)|\mathrm{d}t \\ &\leq \frac{1-r}{1-(1-\delta)r}\int_0^1 |f(t)|\mathrm{d}t + \epsilon.\end{aligned}$$

We conclude that

$$\lim_{r\to 1}(1-r)|(\mathcal{H}f)(re^{i\theta})| = 0, \quad \theta \in [0, 2\pi].$$

Theorem 1 says that $\mathcal{H}f = C\mu$ for some $\mu \in M$ and a simple exercise with the dominated convergence theorem [3, p. 42] says that

$$\lim_{r\to 1}(1-r)(C\mu)(re^{i\theta}) = \mu(\{e^{i\theta}\}).$$

This means that $\mathcal{H}$ maps H^1 to the space of Cauchy transforms of measures on the circle which have no point masses.

Acknowledgements
The author would like to thank the referee for a careful rewriting of the principal result of this paper.

Received: 27 May 2016 Accepted: 7 September 2016
Published online: 16 September 2016

References

1. Duren, P.: Theory of H^p Spaces, Pure and Applied Mathematics **11**. Academic Press, New York (1970)
2. Diamantopoulos, E., Siskakis, A.G.: Composition operators and the Hilbert matrix. Studia Math **140**, 191–198 (2000)
3. Cima, J.A., Matheson, A.L., Ross, W.T.: The Cauchy Transform, Mathematical Surveys and Monographs, **125**. American Mathematical Society, Providence (2006)